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SOLUTION OF THE BENDING PROBLEM OF A CIRCULAR PLATE WITH A FREE EDGE BY USING PAIRED EQUATIONS*

A.A. SAMODUROV and A.S. TIKHOMIROV

Used in the investigation of the bending of a plate with a free edge is a method including the solution of non-standard paired equations that are trigonometric series, which results in a quasi-completely-regular system of linear algebraic equations. The nature of the plate state of stress and strain is investigated. One of the methods reducing to the solution of paired summation equations is elucidated in /1/ in application to the problem under consideration. A combination of support and clamping is examined in the majority of papers while the case of the free edge is inadequately studied.

Let us examine a thin circular plate of radius a and thickness h which is loaded at the center of a circle of radius b by a uniformly distributed load of intensity p. Part of the plate outline is load-free within the angle $2\theta_0$, while the rest is hinge supported (Fig.1).

It is expedient to represent the solution of the initial problem in the form of a sum of two solutions $w_c = w^{(1)} + w^{(2)}$ is the dispalcements, and $\sigma_{ik}{}^c = \sigma_{ik}{}^{(1)} + \sigma_{ik}{}^{(2)}$ in the stresses.

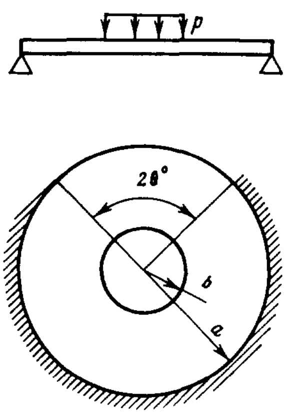

Fig.1

The first problem corresponds to the model of a plate with total support along the outline and the load shown in Fig.1, its solution is known in the literature /2/. The second problem contains the same plate with the contour conditions displayed in Fig.1, but with the load equal numerically to the support reaction q in the first problem but directed oppositely, to the free edge outline. (In this case $q = p\cdot(\pi b^2/2\pi a)$ is the intensity of the uniformly distributed load). Therefore, the solution of the initial problem reduces substantially to an analysis of the second problem, whose boundary conditions for $r = a$ are

$$\begin{aligned} &M_r{}^{(2)} = 0, && 0 \leqslant \theta \leqslant \pi \\ &Q_r^{(2)} - \frac{1}{r}\frac{\partial M_{rt}}{\partial\theta} = P, && 0_0 \leqslant \theta \leqslant \theta_0 \\ &w^{(2)} = 0, && \theta_0 < \theta \leqslant \pi \end{aligned} \tag{1}$$

Here r, θ are the running coordinates, $w^{(2)}$ is the displacement, $Q_r{}^{(2)}$ is the transverse force, and $M_r{}^{(2)}$, $M_{rt}{}^{(2)}$ are the bending moment and torque, respectively.

In connection with the symmetric arrangement of the plate relative to the origin of the angle θ we consider half of it, and we seek the solution in terms of the deflection function $w^{(2)}(r, \theta)$ in the form of a cosine series satisfying the equilibrium equation in polar coordinates

$$\nabla\nabla w^{(2)} = 0$$

On the basis of discussions analogous to those presented in /1/, we obtain the function

$$w^{(2)}(\rho, \theta) = \sum_{n=0}^{\infty} (A_n{}^*\rho^n + B_n{}^*\rho^{n+2})\, a^n \cos n\theta, \quad \rho = \frac{r}{a}$$

in whose terms the fundamental components $M_r{}^{(2)}, M_{rt}{}^{(2)}, Q_n{}^{(2)}$, etc., are expressed /2/, where $A_n{}^*$, $B_n{}^*$ are unknown coefficients. The relationship between the coefficients $A_n{}^*$ and $B_n{}^*$ is found from the first boundary condition. Furthermore, satisfaction of the second and third conditions results in a system of paired summation equations which must be solved for the unknowns A_n. Introducing new unknowns, we write this system in the form (E,ν are the Young's modulus and Poisson's ratio of the plate material)

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$$\sum_{n=2}^{\infty} A_n Y_n \cos n\theta = 1, \quad 0 \leqslant \theta \leqslant \theta_0$$
$$\sum_{n=2}^{\infty} A_n \cos n\theta = -A_0 - A_1 \cos\theta, \quad \theta_0 < \theta \leqslant \pi$$
$$A_n = A_n{}^* a^n \left\{1 - \frac{n(n-1)(1-\nu)}{(n+1)[(n+2)-\nu(n-2)]}\right\} A$$
$$A = -\frac{3+\nu}{12(1+\nu)} \frac{E}{q} \left(\frac{h}{a}\right)^3, \quad Y_n = \frac{n^2(n^2-1)}{2n+(1+\nu)}$$

Therefore, the problem is to seek the unknowns A_n from the system of paired summation equations (2).

Compared to the solutions examined earlier(/13 – 6/, etc.) for the paired summation equations with trigonometric base functions, this problem has a number of singularities: the degrees of the polynomials in n in the equations of the system are distinguished by a quantity greater than one; the first two term are missing in one of the series in the system. Hence, the approach to the solution of the system (2) requires special consideration.

We differentiate the second equation of the system (2) thrice and we write the system obtained in the form

$$\sum_{n=2}^{\infty} D_n \cos n\theta = 1, \quad 0 \leqslant \theta \leqslant \theta_0 \tag{3}$$
$$\sum_{n=2}^{\infty} D_n \sin n\theta = \sum_{n=2}^{\infty} D_n (1-\varphi_n) \sin n\theta - \frac{A_1}{2} \sin\theta, \quad \theta_0 < \theta \leqslant \pi$$
$$\varphi_n = \frac{n^2 + {}^1/_2 n(1+\nu)}{n^2-1}, \quad D_n = A_n Y_n$$

As $n \to \infty$ the quantity $(1-\varphi_n)$ decreases no worse than $O(1/n)$, as is necessary for quasi-complete-regularity of the system (see below). In order to reduce the left sides of the equations in the system (3) to identical form, we multiply the first equation of (3) by $\cos(\theta/2)$ $(\cos\theta - \cos t)^{-1/2}$, and then integrate with respect to θ between 0 and t; we multiply the second equation by $\cos(\theta/2)(\cos t - \cos\theta)^{-1/2}$ and integrate with respect to θ between t and π (this method is described in more detail in /7/). After manipulation, we obtain ($P_n(t)$ are Legendre polynomials)

$$\sum_{n=2}^{\infty} D_n y_n = F(t), \quad 0 \leqslant t \leqslant \theta_0 \tag{4}$$
$$\sum_{n=2}^{\infty} D_n y_n = \sum_{n=2}^{\infty} D_n (1-\varphi_n) y_n - \frac{A_1}{2} y_1 \sin t, \quad \theta_0 < t \leqslant \pi$$
$$y_n = P_{n-1}(\cos t) + P_n(\cos t)$$
$$F(t) = \frac{2\sqrt{2}}{\pi} \int_0^t \frac{\cos(\theta/2)\, d\theta}{\sqrt{\cos\theta - \cos t}}$$

We reduce the system (4) obtained to an infinite system of linear algebraic equations

$$D_n' = \sum_{i=2}^{\infty} \alpha_{ni} D_i' + \beta_n \left(D_n' = 2\frac{D_n}{n}\right) \tag{5}$$
$$\alpha_{ni} = (1-\varphi_i)\frac{i}{2} I_{ni}(\theta_0), \quad \beta_n = \beta_n' - \frac{A_1}{2} I_{n1}(\theta_0)$$
$$\beta_n' = \int_0^{\theta_0} F(t) y_n \operatorname{tg}\frac{t}{2}\, dt, \quad I_{ni}(x) = \int_x^{\pi} y_n y_i \operatorname{tg}\frac{t}{2}\, dt, \quad i = 2, 3, 4, \ldots$$

An unknown quantity is present in the free part of the system (5), and which we find, as we do A_0, from the boundary conditions on the outline ($r = a$)

$$w = 0\ (t = \pi), \quad \int_{\theta_0}^{\pi} w\, dt = 0$$

We obtain

$$A_1 = \sum_{n=2}^{\infty} D_n' \frac{n}{2Y_n} \chi_n \tag{6}$$

$$\chi_n = \left[\frac{\sin n\theta_0}{n(\pi-\theta_0)} + (-1)^n\right]\left(1 - \frac{\sin n\theta_0}{\pi-\theta_0}\right)^{-1}$$

$$A_0 = A_1 - \sum_{n=2}^{\infty} D_n' \frac{n}{2Y_n}(-1)^n$$

Now, if the index of summation n is changed to i in the first expansion of (6), A_1 is substituted in the free term of the system (5), and then the general term of the expansion A_1 and the kernel of the system are combined, we then obtain a system of the following form:

$$D_n' = \sum_{i=2}^{\infty} \alpha_{ni}' D_{ni}' + \beta_n' \tag{7}$$

$$\alpha_{ni}' = \frac{i}{2}\left\{(1-\varphi_i) I_{ni} - \frac{\chi_i}{2Y_i} I_{n1}\right\}$$

Investigation of this system for regularity, analogously to that performed in /7/, shows that it is quasi-completely-regular.

The solution of (7) in the unknowns D_n' was performed by the Gauss method. For this, the degeneracy of the coefficient matrix was estimated, which displayed the good conditionality of the system. Up to 150 unknown terms of the system were kept in the computations, and satisfaction of the boundary conditions of the problem was assured to 2% accuracy. This error evidently diminshes when computing the stress tensor and displacement vector components at remote points from the outline.

Dependence of the radial $\langle\sigma_r^{(2)}\rangle$ and circumferential $\langle\sigma_t^{(2)}\rangle$ components of the stress components on ρ are depicted in Fig.2 (solid and dashed lines, respectively) for $\theta_0 = 20°$ and different values of the angle θ, where

$$\langle\sigma_r^{(2)}\rangle = \frac{6}{h^2}\langle M_r\rangle = \frac{6M_r}{h^2\varkappa}, \quad \langle\sigma_t^{(2)}\rangle = \frac{6}{h^2}\langle M_t\rangle = \frac{6M_t}{h^2\varkappa}, \quad \varkappa = \frac{3pa}{h^2(3+\nu)}$$

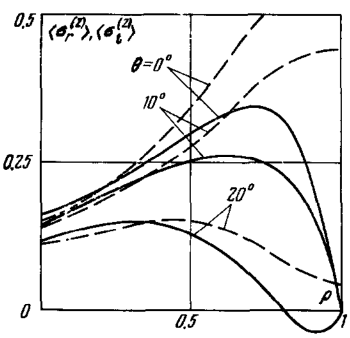

Fig.2

The influence of the magnitude of the free edge on the natures of the plate state of stress and strain is of interest in an analysis of the initial problem. The appropriate estimate can be found from the following relations

$$K_\sigma = \frac{\sigma_r^{(2)}(r=0, \theta=0)}{\sigma_r^{(1)}(r=0)}, \quad K_w = \frac{w^{(2)}(r=0, \theta=0)}{w^{(1)}(r=0)}$$

This estimate yields a representation of the stress relations at the center of the plate for the two problems mentioned. Values of K_w and K_σ are presented below for $b/a = 0.015$; also shown is the nature of the change in the reduced coefficient $A_0 = A_0^* A$ as a function of the angle θ_0 (this coefficient characterizes the magnitude of the deflection at the center of the plate $\rho = 0$)

θ_0, deg.	10	20	30	40
$K_w \cdot 10^4$	151	687	1593	2931
$K_\sigma \cdot 10^4$	24	101	217	359
A_0	0.02	0.06	0.14	0.20

The proposed method of solution can even be extended to the case of arbitrary loading; the principle of constructing the solution remains as before.

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